The Mathematics of Corporate Insurance

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December's article (*Mathematics Today* pages 197-199) "Insurance goes Dating with Science" showed how the insurance industry is making more and more use of complex mathematics, computer modelling and gigabytes of data.

In this article we take an introductory look at some of the challenges facing mathematicians working in the field of corporate insurance and show how corporate insurance is a blend of complex mathematics and practical commercial reality.

The Classical Car Insurance Problem

We will introduce the mathematical concepts with a type of insurance with which most of you will be familiar. Car insurance covers you against injuring third parties or damaging their property (the policy can also be extended to "Comprehensive" cover, covering damage to your own car as well). This type of insurance is an example of "Personal Lines" insurance, i.e. generally sold to individuals. It is not corporate insurance, but is a useful introduction to the mathematics of insurance. A typical example of a car insurance problem might be as shown below:

How much premium would you charge for the following risk?

- 18 year-old male driver
- Drives X-Reg Ford Fiesta, valued at £1,000
- Has 3 penalty points on licence for speeding
- Lives in central London
- Wants third party cover only, for 12 months

Figure 1 The typical car insurance problem

The typical insurance problem may be summarised as follows:

- *Each insured pays a premium (usually different for each insured) to the insurer*
- *The premium is charged in advance and is fixed at the outset of the policy*
- *The insurer will then pay for all damage caused by the insured to third parties (and their property) during the policy period (usually the following 12 months)*

In simple terms, the usual constraints are as follows:

- *We only have very limited data about the "behaviour"/claims history of previous identical (or similar) insureds*
- *The policy may cover very large claims, the size of which have not occurred before for that type of insured*
- *If the premiums are set too high, we will not sell any (or enough) policies; if the premiums are set too low, we could become insolvent, or fall below any minimum "capital adequacy" requirements set by Regulators (as described later)*

For now, we will ignore the need by the insurer to make a profit; we will return to this issue at the end of this article. We will instead focus on the issue of setting equitable premiums across all insureds to cover all (expected) claims.

The Typical Multivariate Solution

This problem is typically solved as a multivariate problem, that is to estimate the claim frequency and claim severity for each combination of factors (such as age of driver, value of car etc) based primarily on the past claims data for similar insureds.

The advent of significant computing power in the 1980s led to the widespread use of the Generalised Linear Interactive Modelling (GLIM) package, using Generalised Linear Models. A seminal Institute of Actuaries paper in 1992 [1] set out the following typical model structure.

Set of data values Yi for i = 1,2, ... n (e.g. claim frequency) for the ith combination of rating factors.

$E(Y_i)= m_i$ and $Var(Y_i) = Q \cdot V(mu_i)/w_i$

mu_i has the form $mu_i = h(z_i)$
h( ) is some known monotonic function
z_i is a linear function of the unknown parameters of the model $_i$
V( ) (the variance function) is known
w_i (the prior weights) are known
the constant factor Q is not necessarily known
the random components of Y_i for i= 1 . . . n are mutually independent.

Figure 2 Structure of Typical Generalised Linear Model (GLM)

Having a modelling framework (and a fast computer!) is all very well, but what to model as predictive factors for claim frequencies and claim severities? When car insurance policies are sold, certain information is collected about the insured (for example, age of driver, make/model of car, number of miles driven, home postcode, number of claim-free years etc.). But some further thought is required before hasty modelling.

Some of these factors may be highly correlated (e.g. young drivers are unlikely to have a high number of claim-free years) whereas some factors may be proxies for "true" predictive factors which have not been collected (e.g. home postcode may be a proxy for affluence, i.e. wealthier people may be less inclined to claim for small amounts – i.e. a lower claim frequency - whereas less affluent people may be more inclined to "get value for money" from their policy). In addition, some factors are virtually (or totally) prohibited from use by Discrimination laws e.g. race, religion, sexual orientation, disabilities etc. Indeed there was recently an attempt by the European Union ("EU") to ban the use of sex as a rating factor, i.e. the EU wanted there to be no difference between premiums charged to a man and a woman driving the same car in the same postcode etc. So the information that you are allowed to collect at the point of sale can change over time, meaning that the predictive factors used in your modelling can change over time.

Over time, some car insurers have introduced more socio-economic rating factors, by asking questions such as marital status, number (and ages) of children, number of cars in household, number of properties owned and whether you are a home owner or a tenant. Of course, unless the insurers had been collecting this information historically at the point of sale, then there is no historic data classified into these categories! In these cases, insurers have to "back-fit" their historic data into these new categories by using other information sources such as census information and wider socio-economic classification databases.

Conversely, some insurers have taken the opposite approach; they want to speed up the sales process by significantly reducing the number of questions. In a real-life case, one corporate insurer client of mine wanted to ask only 2 questions at the point of sale (one of which was "what is your home postcode?", the other question we will leave you to think about!), before offering you a premium for your car insurance. Clearly, in this case, there exists significant potential for cross-subsidies in the premiums, and hence premiums which are out of line with other insurers, leading to “too many” or “too few” policies being sold.

Of course, the “real” predictive factors are unknown; accidents are often caused by complex human interactions which may not be predictable using current data; there is even talk of the process of genetic testing one day identifying a human “risk-taking” gene, thus “predicting” who is likely to be a “good” or “bad” driver; such genetic tests could become a requirement for a discount on your car insurance although this is certainly many years, perhaps decades, into the future. So there is uncertainty about the predictive power of any GLM for car insurance; this uncertainty is often modelled using simulation techniques, parameter confidence intervals, bootstrapping and Bayesian methods, which are all used with the aim of attempting to quantify the uncertainty, and hence charging an appropriate premium to reflect the level of uncertainty.

Commercial Lines (“Corporate”) Insurance

We will now turn to corporate insurance policies; these are sold to companies usually covering multiple risks (e.g. Motor Fleet insurance, covering many cars under the same policy). Such policies are also known as “Commercial Lines” policies.

The total assets (and hence the potential sizes of claim) of a single global corporation can easily exceed £1 billion, which exceeds the underwriting capacity of any single insurance company. Therefore, the coverage must be shared between a number of insurance companies. Some insurers will only be interested in certain aspects of the risk (e.g. fine art collections, offshore oil rigs, factories etc.). Each potential insurer must decide whether or not it wishes to underwrite part of the risk and, if so, at what premium and for what percentage (of the Total Insured Value) it wishes to cover.

It is not uncommon for as many as 50 insurers to share such a £1 billion risk between them (with each insurer typically taking its own individual share – it is not as easy as each insurer just taking 1/50th of the risk!). Put another way, each insurer will have its own mathematical model of the risk, leading to a different premium for the same risk. Corporate insurance premiums are therefore often subject to intense negotiations.

Typically, companies will accept an “excess” of anything from c£50,000 per claim to £50million (or more) per claim, depending upon their balance sheet strength; in other words, the insurer only pays for claims above these high amounts, i.e. the distribution of claims to the insurer can be severely truncated. In the extreme (real-life) case of a £50million excess per claim, the insurer will only pay out for claims over £50million; this is typically known as CAT (“Catastrophe”) cover and presents considerable mathematical challenges.

Commercial lines policies thus add several layers of complexity to the mathematics of corporate insurance; these complexities are summarised in Table 3.

Table 3: Personal Lines Insurance vs Commercial Lines Insurance

Personal Lines Insurance	Commercial Lines Insurance
Premiums typically £100-£2,000	Premiums typically £1 million-£50 million
Policy excess typically £100-£1,000 per claim; insurer’s claims distribution only mildly truncated	Policy excess typically £50,000-£50million (or more) per claim; insurer’s claims distribution severely truncated and may cover “catastrophic” claims only (CAT cover).
Maximum claim from a single insured probably c£10 million	Maximum claim from a single insured probably c£10 billon+ (e.g. oil pollution, catastrophic explosion, release of toxic/radioactive materials etc)
Usually single risks	Multiple risks in many countries, constantly changing over time (e.g. property acquisitions and disposals, property re-valuations, currency movements)
Usually relatively small risks – any one insured typically does not dominate the insurance market for that type of risk	A single global corporation can easily own more than £1 billion of assets, which exceeds the underwriting capacity of any single insurer
Risks relatively homogeneous	Very diverse risks (e.g. ranging from office blocks to nuclear installations, for a single insured)
Whole risk usually placed with one insurer	Whole risk can be placed with as many as 50 different insurers
The proposal form (the description of the risk or the “Risk Profile”) is typically quite straightforward, perhaps 3-4 pages at most	The Risk Profile can be highly detailed, perhaps 100-200 pages for a global corporation
Commercial lines insurances present many more challenges to mathematicians	

Why Multivariate Methods Usually Fail

The obvious approach is to extend the multi-variate (GLM) approach (as almost universally used in all types of personal lines insurance e.g. car insurance, home insurance, travel insurance etc.) to the more complex risks of corporate insurance. Whilst GLM approaches are gaining popularity with increasing computing power, there are features of corporate insurances which make such methods difficult to use reliably. We will illustrate these difficulties using 3 different types of corporate insurance, namely Property Catastrophe (CAT) Cover, Medical Malpractice and Employer’s Liability.

Property Catastrophe (CAT) Cover

Property CAT typically covers property claims in the extreme tail of the underlying distribution; in the real-life case noted earlier of a £50million excess per claim, the insurer is providing cover of (say) up to £1billion but only for claims over £50million. Such claims can easily be caused by earthquakes, tsunamis, explosions etc, and are clearly extremely rare. Put another way, there is very little historic claims data for events of this size, thus GLM approaches fail due to the extreme sparsity of data within the homogeneous factor sub-groups.

Probability distributions for such rare events are often modelled using Generalised Pareto Distributions (GPDs) or Generalised Extreme Value (GEV) distributions fitted to the very limited historic data. GPDs and GEVs are an evolving area of mathematical research; inevitably, as more rare incidents occur, the GPDs and GEVs are refined to produce more accurate results.

As a practical example, GEVs can be used to predict very approximately where and when earthquakes might be due. GEVs suggest a theoretical maximum intensity of around 8.6 [2] on the Richter scale. This theoretical maximum is supported by current geophysical evidence that earthquakes occurring above this level would release so much localized energy that plastic, rather than brittle, deformation of the surrounding rocks would be caused. Thus current geophysical evidence supports the idea that GEV models are of some benefit in forecasting localized earthquake magnitudes.

Figure 4 shows some generalised GPD and GEV functions.

Generalised Pareto Distribution (GPD)

$P(X < x|X > u) =$

$1-[1+g(x-m)/s]^{(-1/g)}$ for $g <> 0$

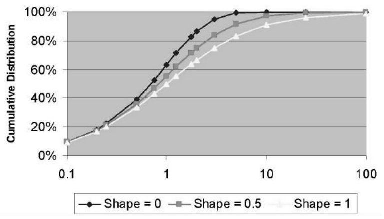
$1-\exp[(-x-m)/s]$ for $g = 0$

m = location

s = spread

g = shape

u = threshold



Generalised Extreme Value (GEV) Distribution

$P(Y < y) = GEV(y; g, \mu, s) =$

$\exp(-[1+g(y-m)/s]^{(-l/g)})$

Estimate y_p where $GEV(y_p) = 1-p$

y_p is return level associated with return period $1/p$

μ is location parameter

s is the scale parameter

g is the shape parameter

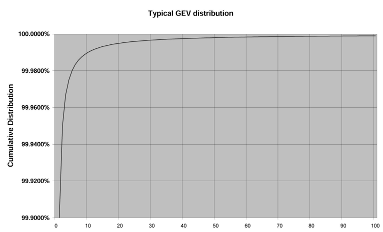


Figure 4 Tools for Extreme Values

Medical Malpractice Cover

This type of cover is usually bought by hospitals (or Hospital Trusts in the UK), to provide compensation in the event of an "unwanted medical outcome" due to a surgical mistake. In other words, if a surgeon errs negligently (which leads to pain, suffering etc for the patient) the hospital's insurance policy will pay compensation to the patient.

Sparsity of claims is not a problem for this type of cover – recent figures from the NHS show that £560 million was paid out in negligence claims against the NHS in 2005/06 [3].

The main problem is the very high risk surgeons; it may be surprising to know that brain surgery, whilst inherently high risk, is not the main cause of insurers' problems for this type of cover; since such surgery is inherently high risk, and it is usually the patient's choice to take this high risk, claims for negligence for brain surgery are relatively rare.

The biggest problem for insurers of this type of risk is obstetricians (doctors who specialise in pregnancy and childbirth) and also midwives [4], for the following reasons:

- *Whatever the medical risk, the patient usually has no option but to have the baby delivered at some point*
- *A surgical mistake during a birth can result in a highly brain damaged baby; the costs of caring for such a child for the rest of its' life (ie. the compensation payable by the insurer) usually runs into millions of pounds, over perhaps 20-30 years*
- *Such "mistakes" can take many years after the birth to be fully identified; in many cases it is not until the child turns 18 years of age that a claim is finally made against the surgeon, i.e. 18 years after the surgical procedure*
- *Medical malpractice policies usually cover surgeons on a so-called "claims made" basis, i.e. they are covered for any claims made against them today, even if the underlying surgery took place many years earlier. Thus, for obstetricians, the insurer today is at risk of claims being made for surgical procedures carried out by that surgeon over the past 20 years or so (even if the surgeon was insured by a different insurer at the time of the surgery)*

Due to the very long reporting delays for these types of claims and the very long period of time over which claims are paid, complex GLM models are out of place.

There is therefore tremendous uncertainty both in claim frequencies and claim severities for insurers of such surgeons. This is reflected in extremely high premiums, "alternative" government schemes for public sector surgeons (e.g. the UK NHS Litigation Authority, "the NHSLA") and the development of "no fault" compensation schemes in other countries.

Employer's Liability

This type of cover must be bought by all employers to provide compensation in the event of an employee having an accident at work due to an employer's negligence.

The main problem is asbestos-related claims, and in particular, mesothelioma claims, which have caused the downfall of a number of corporate insurers. From the onset of mesothelioma, an unpleasant death usually follows within 12 months, with little hope of improvements in life expectancy in the future.

The annual number of mesothelioma deaths has increased considerably over the period for which statistics are available, reaching 1969 deaths in 2004, the latest year for which data are available, compared with only 153 deaths in 1968. The latest projections [5] suggest that the annual number of mesothelioma deaths in Great Britain will peak at around 1950 to 2450 deaths some time during the period 2011 to 2015.

However these claims have generally arisen out of periods of employment in the 1950s and 1960s. During these years, asbestos was seen as the "wonder fibre", with amazing fire retardant properties which led to it being used extensively for fire protection, for example in the ship-building industry. It was only in the mid-1960s that the health effects began to be identified and it was not until the early 1970s that restrictions were put on its use.

Mesothelioma is the only asbestos-related disease which can remain dormant in the body for at least 30 years, from which point the disease can develop over the next 10 years or so; typically employees who were "exposed" to asbestos dust in their 20s do not develop the disease until their 60s or 70s. In other words insurers in the 1950s and 1960s are today facing claims from employers they insured perhaps as long as 50 years ago. Mesothelioma claims usually cost at least £100,000 each, due to the unpleasantness of the disease, and there are usually complex

Predicted number of mesotheliomas at age A in year $T(F_{AT})$

$$F_{A,T} = \left[\sum_{l=0}^{A+1} W_{A-l} D_{T-l} \{l + 1 - L\}^k 0.5^{\frac{l}{H}} \right] Dx_T P_{A,T} \frac{M}{F}$$

Where

$P_{A,T}$ person years for age A in year T;

D_T overall population exposure in year T;

Dx_T proportion of occurring mesotheliomas diagnosed in year T;

W_A age specific exposure potential at age A;

L lag period (in years) before effect starts;

H half life (in years) for clearance of asbestos from lungs;

k exponent of time modelling increase of risk with increasing time from exposure;

M total observed mesotheliomas;

The content of the { } is set to zero when negative; and

$$F = \sum_{A,T} \left[\sum_{l=0}^{A+1} W_{A-l} D_{T-l} \{l + 1 - L\}^k 0.5^{\frac{l}{H}} \right] Dx_T P_{A,T}$$

Figure 5 HSE model for mesothelioma claims

legal arguments over the precise meaning of the policy wordings, claim sharing between different employers and so on.

There have been various mathematical/epidemiological models over the years to try to forecast future claims, an example of which is the current Health & Safety Executive model shown above [6].

Thus it can take c40 years for all asbestos-related claims to be reported from a given year of insurance cover; indeed it is a current statutory requirement for employers to keep copies of their employer's liability insurance certificates for 40 years for precisely this reason. Although asbestos is not used to the same extent today, insurers are wary of other similar "dormant" diseases which may be caused by employment conditions; perhaps work-related stress, repetitive strain injury, "sick building syndrome", wireless offices etc etc will be the "new" asbestos problem for insurers in 40 years' time?

Therefore, as it can take perhaps decades for all claims to be reported from a given year of insurance, complex GLM models are out of place.

Making a Profit

Insurers operate in the commercial world; they are risk-takers, charging premiums to cover unknown, but potentially very large (potentially up to £billions) future events but they are ultimately in business to make a profit. In general, insurers can charge whatever premiums they like, there is no regulatory intervention (in the UK at least) setting minimum or maximum levels of premium for a given policyholder. Various complex models of return on capital, expenses and investment returns are used to develop profit margins to add to premiums, and these are often modelled stochastically, due to the long claim payment delays.

Of course, from the insurer's perspective, the profit on a single policy is maximised if it charges a high premium, but, due to competition, not many policies may be sold at that price. Conversely premiums which are too low could threaten the insurer's capital adequacy and hence its' approval from the regulator to trade.

The insurance regulator in the UK (the Financial Services Authority, FSA) sets minimum levels of capital each insurer must hold on an insurer-by-insurer basis in order to allow the insurer to continue trading; capital is essentially the "spare" cash the insurer has after meeting its expected liabilities. These Individual Capital Adequacy Standards ("ICAS") have developed in sophistication over the past few years; current standards are based on 99.5th percentiles of complex joint distributions. In other words, insurers must hold sufficient capital to remain solvent over a 12-month period at the 99.5th percentile of the joint aggregate distribution across all of its assets and liabilities.

Copulas are of immense value in estimating the overall joint distributions; they are a simpler mathematical form of multivariate probability distributions and contain the whole information about the variables' dependency structure.

Actual Premiums

The definition of a copula is a function $C: [0,1]^N \rightarrow [0,1]$ where:
(a) there are random variables $U_1, \ldots, U_N$ taking values in [0, 1] such that C is their distribution function; and
(b) C has uniform marginal distributions, i.e. for all $i \le N$, $u_i \in [0,1]$, we have: $C(1 \ldots 1, u_i, 1 \ldots 1) = u_i$.

The basic rationale for copulas is that any joint distribution F of a set of random variables $X_1, \ldots, X_N$,
i.e. $F(x) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_N \le x_N)$,
can be separated into two parts. The first is the marginal distribution functions, or *marginals,* for each random variable in isolation, i.e. $F_i(.)$ where $F_i(x) = P(X_i x)$. The second is the *copula* that describes the dependence structure between the random variables. Mathematically, this decomposition relies on Sklar's theorem, which states that, if $X_1, \ldots, X_N$ are random variables with marginal distribution functions $F_1, \ldots, F_N$ and joint distribution function F, then there exists an N–dimensional copula C such that, for all $\mathrm{x} \in \Re^N$:

$$F(\mathrm{x}) = C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)) = C(\mathbf{F}(\mathbf{x}))$$

i.e. C is the joint distribution function of the unit random variables $(F_1(x_1), F_2(x_2), \ldots, F_N(x_N))$. If $F_1, \ldots, F_N$ are continuous, then C is unique.

Figure 6 Copulas

In the world of insurance, having a (probabilistic) range of premiums is no good; we must quote a single premium for the risk and generally the premium is fixed once quoted, so we do not have a second chance if we are wrong.

The actual real-life premiums in the examples above depend on a huge amount of detailed risk information which is not shown here for brevity, but to give you an idea, here are "typical" annual premiums for the risks we have looked at:

- *18-yr old driver with 3 penalty points: £1,000 - £5,000* [7]
- *Property CAT: £1bn of cover for claims over £50m: £10million - £20million*
- *Medical Malpractice cover: £1,000 - £200,000 per surgeon*
- *Employer's Liability: £50 - £500 per employee*

Summary

Insurers are risk-takers, accepting premiums to cover unknown, but potentially very large, future insured events. Insurance is a finite resource, and therefore comes at a "price". Due to the complexity and size of corporate risks, the very limited historic claims data for very large claims and the need to estimate extreme tails of complex joint distributions to satisfy regulators, the mathematics of corporate insurance is especially challenging. However, insurers have to balance the theoretical mathematics with commercial considerations; they must be profitable but yet offer commercially acceptable premiums whilst at the same time satisfying minimum regulatory capital requirements.

New risks are emerging all the time; for example, telecommunications companies often require "Business Interruption" insurance in the event of network failure causing them to pay compensation to their customers. The recent IMA conference "The Mathematics of Signal Processing" gave useful insights into the mathematics of network reliability, signal quality etc for this emerging area of risk. Future IMA conferences on "Industrial Reliability" and "Flood Modelling" are also of relevance for "Business Interruption" coverage for large industrial operations and "Commercial Property" insurance for companies with operations located in flood prone areas.

In addition, with the relatively recent "compensation culture", new legal precedents and increasingly complex regulatory capital requirements, even "old" risks are not as simple as they used to be!❑

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letters

'Brittle knowledge'

I just wanted to congratulate David Broomhead on his April Editorial.

As the last of Fergus Campbell's research students, I know that he would have really appreciated the comments on the Campbell and Robson paper. He would also have been 100% in support of the remarks about not teaching just to enable students to pass an exam. His view was that with excellent teaching, exams became largely irrelevant.

Sadly, teaching is not uniformly excellent, especially in mathematics. Fear is indeed widespread, not just among students: it's a barrier to progress in every field where the ability to think creatively is required. Thinking itself implies a risk, of course, which is often too great for institutions to contemplate, so they follow the set procedures and achieve guaranteed mediocrity – 'brittle knowledge' as Feynman called it.

Teaching seems so rulebound now that trying to explain anything not strictly required for some exam is often impossible. Similarly, schools are unable to make use of my modest attempts to volunteer to help teach maths and physics -because I'm not a 'qualified teacher.'

We are letting young people down badly by conforming to rules which limit real education.

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Continual Emphasis

I was very interested to read the Editor's comment in the April issue of *Mathematics Today* on the fear of mathematics among his optometry students.

It seems to me this is inherent in the nature of the subject. Mathematicians enjoy the subject as they model a problem and use their skills and ability to find an answer. This means it is fair to test trainee mathematicians with some examination questions, but not all, that need ingenuity in their solution. However, such questions are not suitable for anyone who only needs to follow a presented mathematical argument. Such students can be conscientious in their studies, but when it comes to problems they often do not know how to begin. Hence the fear and frustration as they know they have tried to study diligently. Mathematics is very different to many other subjects in this respect. If for example a history student has studied revolutions then they know for certain that something can be written about any particular event. Of course they may find it extremely difficult to give a good answer if asked to contrast the English, American, French and Russian revolutions. So history has its own difficulties, but the student can do something. This is not always possible with a mathematics question, particularly in a limited time. So these students need very carefully constructed examination questions, which might include a guided solution of a problem. The illustration of $-2(-1)^n = 2(-1)^{n+1}$ is perfectly reasonable in a clearly designated question on indices. Teaching should be directed towards reading mathematics and as the students gain the ability to understand this language so they should get satisfaction.

I have no doubt these comments are very familiar, but they need continual emphasis.❑

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